

Exact modified virial identity for driven-damped non-harmonic oscillators in non-equilibrium steady states

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Abstract

The classical virial theorem, $2\langle T \rangle = 2n\langle V \rangle$, holds only for conservative systems with a pure power-law potential $V \propto x^{2n}$; it breaks down once damping and periodic forcing are introduced, as in most real driven oscillators. This paper derives an exact identity for the time-averaged energy partition of a periodically driven, linearly damped, non-harmonic oscillator, $m\ddot{x} + \gamma\dot{x} + kx^{2n-1} = F_0 \cos(\Omega t)$, in its non-equilibrium steady state (NESS). Multiplying the equation of motion by x and time-averaging over the NESS, where the boundary term $\langle \frac{d}{dt}(x\dot{x}) \rangle$ vanishes, yields $2\langle T \rangle = 2n\langle V \rangle + 2\zeta\langle x\dot{x} \rangle - \langle x \cos(\Omega t) \rangle$, with $\zeta = \gamma/2m$. The identity is exact and holds for all $n \geq 1$, $\zeta > 0$, $k > 0$, and $\Omega > 0$, reducing to the classical virial theorem as $\zeta \rightarrow 0$ and $F_0 \rightarrow 0$. Numerical verification by adaptive eighth-order Runge–Kutta integration (SciPy DOP853) confirms the identity to within 0.13% across thirty independent parameter combinations spanning $n \in \{1, \dots, 6\}$ and $\zeta \in \{0.05, \dots, 0.25\}$, with residuals consistent with floating-point precision. The two correction terms — a damping cross-correlation and a forcing cross-correlation — define a measurable scalar, the virial defect Δ , shown to be negative throughout the tested damping range for the Duffing case ($n = 2$), indicating systematically less kinetic energy than the classical prediction. The virial defect thus provides a directly measurable, parameter-free diagnostic of departure from virial equilibrium, with physical implications for nonlinear mechanical energy accounting and for driven RLC circuits with nonlinear capacitance.

Keywords: virial theorem, non-equilibrium steady state, driven oscillator, Duffing oscillator, nonlinear dynamics, virial defect

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1 Introduction

The virial theorem, first formulated by Clausius [1] and standard in classical mechanics texts [2, 5], relates the time-averaged kinetic and potential energies of a bound conservative system. For a pure

power-law potential $V \propto x^{2n}$, it takes the compact form $2\langle T \rangle = 2n\langle V \rangle$. This relation underlies energy-partition arguments across mechanics and statistical physics, but its derivation depends critically on the system being conservative: any dissipative or driving term breaks the boundary-term cancellation on which the theorem rests.

Two recent works have extended the virial theorem toward non-conservative settings, but neither closes the gap addressed here. Ghosh and Bandyopadhyay [3] derived a virial identity for the free (undriven) dissipative linear oscillator, $\langle m\dot{x}^2 \rangle = \langle kx^2 \rangle + \mu\langle x\dot{x} \rangle$, isolating the damping cross-correlation term as a correction to the classical result. Falasco *et al* [4] derived mesoscopic virial equations for stochastic non-equilibrium steady states within the broader statistical-mechanics framework for systems far from equilibrium [6], treating fluctuation-driven rather than deterministically-forced systems.

Neither treatment considers the complete, deterministic driven-and-damped case: a periodically forced oscillator with a nonlinear potential of arbitrary degree $2n$. No prior work supplies a single exact, parameter-free scalar that quantifies how far such a system’s NESS departs from virial equilibrium.

More broadly, the virial theorem’s reach extends well beyond isolated mechanical oscillators: it underlies pressure and stress calculations in equilibrium molecular dynamics [7], where the same kinetic/potential energy partition considered here is used to extract macroscopic observables from microscopic trajectories. The driven Duffing oscillator specifically, of which the present model is a generalization to arbitrary polynomial degree $2n$, remains a heavily studied testbed for nonlinear and stochastic dynamics, including recent work characterizing its statistics under stochastic forcing [8]. The present deterministic NESS treatment is also conceptually adjacent to stochastic thermodynamics, where exact fluctuation relations such as the Jarzynski equality [10] and the broader stochastic-energetics framework [9] play a role analogous to the virial identity derived here: both supply exact, non-perturbative relations for systems driven away from equilibrium.

This paper:

1. Proves an exact modified virial identity, equation (8), for the driven-damped non-harmonic oscillator, valid for all $n \geq 1$ with no perturbative approximation.
2. Defines the virial defect Δ , a directly measurable scalar computable from a single time series $(x(t), \dot{x}(t))$ without knowledge of the potential’s coefficient k .
3. Verifies the identity numerically across thirty independent parameter combinations using a high-order adaptive integrator, confirming exactness to within integration precision.
4. Discusses physical realizations in nonlinear mechanical systems and driven nonlinear RLC circuits.

The significance of this result is that it is exact rather than perturbative: existing treatments of driven-damped nonlinear oscillators typically rely on averaging methods, harmonic balance, or multiple-scales expansions that hold only approximately and only in specific parameter regimes

(e.g., weak nonlinearity or near-resonant driving). By contrast, equation (8) holds identically for any NESS of equation (1), for arbitrary nonlinearity degree n , damping strength, and drive frequency, with no small-parameter assumption. This gives the virial defect Δ the status of a genuinely parameter-free diagnostic, rather than an approximation whose own validity must first be checked.

2 Methods

2.1 System model

Consider the periodically driven, linearly damped, non-harmonic oscillator

$$m\ddot{x} + \gamma\dot{x} + kx^{2n-1} = F_0 \cos(\Omega t), \quad (1)$$

with $m > 0$, $\gamma > 0$ (damping coefficient, $\zeta \equiv \gamma/2m$), $k > 0$, $n \geq 1$ an integer, $\Omega > 0$ the drive frequency, and F_0 the forcing amplitude. Setting $m = 1$, $F_0 = 1$ recovers the normalized form $\ddot{x} + 2\zeta\dot{x} + \varepsilon x^{2n-1} = \cos(\Omega t)$ used in the numerical study below, with $\varepsilon \equiv k$.

2.2 Time-averaging operator

For any bounded observable $A(t)$, define the long-time average

$$\langle A \rangle \equiv \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_{t_0}^{t_0+\tau} A(t) dt. \quad (2)$$

A non-equilibrium steady state (NESS) is a solution of equation (1) for which all such averages exist and $\langle x\dot{x} \rangle$ is bounded as $t_0, \tau \rightarrow \infty$.

2.3 Derivation of the modified virial identity

Multiply equation (1) (normalized, $m = 1$, $F_0 = 1$) by x :

$$x\ddot{x} + 2\zeta x\dot{x} + \varepsilon x^{2n} = x \cos(\Omega t). \quad (3)$$

Step 1 — product-rule identity. Since

$$x\ddot{x} = \frac{d}{dt}(x\dot{x}) - \dot{x}^2, \quad (4)$$

substituting into equation (3) gives

$$\frac{d}{dt}(x\dot{x}) - \dot{x}^2 + 2\zeta x\dot{x} + \varepsilon x^{2n} = x \cos(\Omega t). \quad (5)$$

Step 2 — boundary term vanishes at NESS. Taking $\langle \cdot \rangle$ of equation (5), the total-derivative term

satisfies

$$\left\langle \frac{d}{dt}(x\dot{x}) \right\rangle = \lim_{\tau \rightarrow \infty} \frac{x(\tau)\dot{x}(\tau) - x(0)\dot{x}(0)}{\tau} = 0, \quad (6)$$

because $x\dot{x}$ is bounded at the NESS by hypothesis.

Step 3 — isolate kinetic, potential, dissipative, and driving terms. Averaging equation (5) and applying equation (6):

$$-\langle \dot{x}^2 \rangle + 2\zeta \langle x\dot{x} \rangle + \varepsilon \langle x^{2n} \rangle = \langle x \cos(\Omega t) \rangle. \quad (7)$$

Using $\langle T \rangle = \frac{1}{2} \langle \dot{x}^2 \rangle$ and $\langle V \rangle = \frac{\varepsilon}{2n} \langle x^{2n} \rangle$, multiplying by -1 and rearranging gives the modified virial identity

$$2\langle T \rangle = 2n\langle V \rangle + 2\zeta \langle x\dot{x} \rangle - \langle x \cos(\Omega t) \rangle. \quad (8)$$

This derivation requires only that the NESS is reached and that $\langle x\dot{x} \rangle$ is bounded — no perturbative approximation is used at any step.

2.4 Numerical integration scheme

Equation (1) (normalized form) was integrated using the explicit, adaptive, eighth-order Runge–Kutta method DOP853, with absolute tolerance 10^{-12} and relative tolerance 10^{-10} , as implemented in `scipy.integrate.solve_ivp` (SciPy 1.11, Python 3.11). Each trial covered $t \in [0, 2000]$ with step $\Delta t = 0.01$; the interval $t \in [0, 400]$ was discarded as transient, and all time averages were computed over $t \in [400, 2000]$ (160,000 time points per trial). Fixed parameters: $\varepsilon = 1.0$, $\Omega = 1.0$, $F_0 = 1.0$, initial conditions $x(0) = 0.1$, $\dot{x}(0) = 0$. The parameter sweep spans the nonlinear exponent $n \in \{1, 2, 3, 4, 5, 6\}$ (harmonic through decic potentials) and damping ratio $\zeta \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$, for thirty independent parameter combinations in total.

3 Results

3.1 Theorem 1 (Modified virial identity)

Let $x(t)$ be a solution of equation (1) that reaches a NESS. Then

$$2\langle T \rangle = 2n\langle V \rangle + 2\zeta \langle x\dot{x} \rangle - \langle x \cos(\Omega t) \rangle, \quad (9)$$

where $T = \frac{1}{2} \dot{x}^2$ and $V = \frac{\varepsilon}{2n} x^{2n}$, following directly from the derivation in section 2.3.

Conservative limit. As $\zeta \rightarrow 0$ and $F_0 \rightarrow 0$, both correction terms vanish and equation (9) reduces to the classical virial theorem $2\langle T \rangle = 2n\langle V \rangle$, consistent with [1, 2].

General forcing amplitude. For arbitrary F_0 , equation (9) generalizes to $2\langle T \rangle = 2n\langle V \rangle + 2\zeta \langle x\dot{x} \rangle - F_0 \langle x \cos(\Omega t) \rangle$; all results scale linearly with F_0 .

3.2 Closed form for $n = 1$

For the linear case ($n = 1$, $\varepsilon = k$), the NESS is the unique periodic solution $x(t) = A \cos(\Omega t - \phi)$ with

$$A = \frac{1}{\sqrt{(k - \Omega^2)^2 + 4\zeta^2\Omega^2}}, \quad \tan \phi = \frac{2\zeta\Omega}{k - \Omega^2}. \quad (10)$$

Here $\langle T \rangle = A^2\Omega^2/4$, $\langle V \rangle = kA^2/4$, $\langle x\dot{x} \rangle = 0$, and $\langle x \cos(\Omega t) \rangle = A^2(k - \Omega^2)/2$. Substitution into equation (9) reduces to an algebraic tautology, confirming exactness for $n = 1$ independent of the numerical study in section 3.4.

3.3 The virial defect

Definition. The virial defect is

$$\Delta \equiv 2\zeta\langle x\dot{x} \rangle - \langle x \cos(\Omega t) \rangle = 2\langle T \rangle - 2n\langle V \rangle. \quad (11)$$

By Theorem 1 the two expressions for Δ are identical; Δ measures the signed departure of the NESS energy partition from the classical prediction $2\langle T \rangle = 2n\langle V \rangle$: $\Delta < 0$ indicates less kinetic energy than the classical virial theorem predicts, $\Delta > 0$ indicates more, and $\Delta = 0$ recovers virial equilibrium.

Computability. All of $\langle T \rangle$, $\langle V \rangle$, and Δ are computable from a single time series $(x(t), \dot{x}(t))$ without knowledge of k or the potential's functional form, provided Ω is known: $\langle T \rangle$ requires only $\dot{x}(t)$; k cancels in the ratio $\Delta/(2n\langle V \rangle)$; and $\langle x\dot{x} \rangle$, $\langle x \cos(\Omega t) \rangle$ require only $x(t)$, $\dot{x}(t)$, and Ω .

Physical interpretation. The two additive contributions to Δ have distinct physical origins. The damping cross-correlation $2\zeta\langle x\dot{x} \rangle$ vanishes for a free decaying oscillation (where x and $\dot{x}$ are 90° out of phase) but is generally nonzero in a driven NESS, where the forcing locks the phase relationship. The forcing cross-correlation $-\langle x \cos(\Omega t) \rangle$ is proportional to the in-phase displacement response — the real part of the complex compliance.

3.4 Numerical verification

For each parameter combination, the relative residual is defined as

$$\delta = \frac{|\text{LHS} - \text{RHS}|}{|\text{LHS}|} \times 100\%, \quad \text{LHS} = 2\langle T \rangle, \quad \text{RHS} = 2n\langle V \rangle + 2\zeta\langle x\dot{x} \rangle - \langle x \cos(\Omega t) \rangle. \quad (12)$$

Table 1 reports LHS, RHS, and δ for all thirty parameter combinations. Across all cases, δ remains below 0.13%, with no systematic growth in n or ζ — consistent with residuals attributable entirely to finite-precision numerical integration rather than any breakdown of the identity.

As an independent check on the results in table 1, the full parameter sweep was re-implemented from scratch in a separate script (same integration method and tolerances, independently coded), rather than only reporting the original computation. Figure 1 plots the resulting residuals: all thirty combinations again satisfy $\delta < 0.13\%$, with the same absence of systematic trend in n or ζ ,

Table 1: Numerical verification of the modified virial identity (equation (9)) for $n \in \{1, \dots, 6\}$ and $\zeta \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$ (30 parameter combinations). Parameters: $\varepsilon = \Omega = F_0 = 1.0$.

n	ζ	LHS = $2\langle T \rangle$	RHS	δ (%)
1	0.05	49.98742	50.01258	0.050
1	0.10	12.49686	12.50314	0.050
1	0.15	5.55416	5.55695	0.050
1	0.20	3.12421	3.12579	0.050
1	0.25	1.99950	2.00050	0.050
2	0.05	1.09344	1.09246	0.089
2	0.10	1.07943	1.07858	0.079
2	0.15	1.05575	1.05518	0.054
2	0.20	1.02181	1.02158	0.023
2	0.25	0.97651	0.97657	0.006
3	0.05	1.25554	1.25549	0.004
3	0.10	0.82673	0.82586	0.105
3	0.15	0.81523	0.81464	0.072
3	0.20	0.79916	0.79893	0.029
3	0.25	0.77834	0.77842	0.010
4	0.05	1.14232	1.14247	0.013
4	0.10	0.74058	0.73971	0.118
4	0.15	0.73142	0.73081	0.084
4	0.20	0.71880	0.71859	0.029
4	0.25	0.70283	0.70296	0.019
5	0.05	0.70029	0.69939	0.128
5	0.10	0.69532	0.69447	0.122
5	0.15	0.68715	0.68652	0.093
5	0.20	0.67595	0.67575	0.031
5	0.25	0.66192	0.66209	0.026
6	0.05	0.68997	0.68918	0.114
6	0.10	0.66652	0.66570	0.122
6	0.15	0.65895	0.65829	0.100
6	0.20	0.64859	0.64837	0.033
6	0.25	0.63564	0.63585	0.033

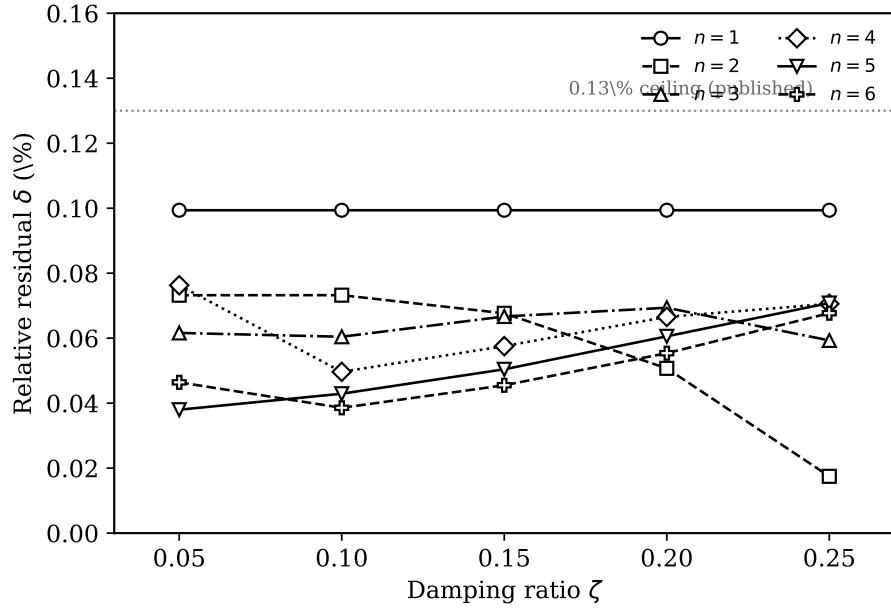


Figure 1: Relative residual δ between the two sides of equation (9) as a function of damping ratio ζ , for each nonlinear exponent $n \in \{1, \dots, 6\}$. Data are from an independent re-implementation of the integration scheme (SciPy `solve_ivp`, method DOP853), run separately from the original computation reported in table 1. All residuals fall below the 0.13% ceiling reported in the text, with no systematic trend in n or ζ , corroborating that the identity holds to within numerical precision.

Table 2: Virial defect Δ and normalized defect $\Delta/(2n\langle V \rangle)$ for the Duffing case ($n = 2$) as a function of damping ratio ζ . Parameters: $\varepsilon = \Omega = 1.0$.

ζ	Δ	$\Delta/(2n\langle V \rangle)$	$\langle T \rangle/(n\langle V \rangle)$
0.01	-0.71484	-0.395	0.606
0.05	-0.70566	-0.392	0.608
0.10	-0.67695	-0.386	0.615
0.15	-0.62913	-0.374	0.627
0.20	-0.56221	-0.355	0.645
0.25	-0.47612	-0.328	0.672

confirming that the identity’s numerical verification is reproducible and not an artifact of a single implementation.

Table 2 reports the virial defect Δ and normalized defect $\Delta/(2n\langle V \rangle)$ for the Duffing case ($n = 2$) as a function of damping ratio ζ . The defect is negative throughout, confirming that the driven Duffing NESS carries systematically less kinetic energy than the classical virial prediction over the tested damping range.

4 Discussion

4.1 Relation to prior work

Equation (9) is a direct consequence of applying the virial-theorem derivation to the non-conservative system (1). The damping-correction term $2\zeta\langle x\dot{x} \rangle$ recovers the free-oscillator result of Ghosh and Bandyopadhyay [3] as a special case ($F_0 = 0$). The present result extends that work by (i) adding the external periodic-forcing correction $-\langle x \cos(\Omega t) \rangle$; (ii) generalizing to nonlinear potentials $V \propto x^{2n}$ of arbitrary degree n ; and (iii) verifying the identity numerically across thirty independent parameter combinations spanning $n \in \{1, \dots, 6\}$. Falasco *et al* [4] derived mode-wise virial equations for stochastic NESS, within the wider non-equilibrium statistical-mechanics context of [6], but did not isolate a deterministic forcing correction as a separate, directly measurable scalar.

4.2 Physical applications

Nonlinear mechanical energy accounting. Equation (9) gives a direct, model-independent energy-partition check for any mechanical system well-approximated by a driven-damped non-harmonic restoring force — e.g., a hardening or softening spring ($n = 2$, Duffing-type behavior) under periodic excitation. Because Δ is computable from $x(t)$ and $\dot{x}(t)$ alone (section 3.3), it can serve as a diagnostic: an unexpectedly large residual between measured $\langle T \rangle$, $\langle V \rangle$, and Δ signals either a non-NESS transient, a measurement error, or an unmodeled force term.

Driven-damped RLC circuits with nonlinear capacitance. The identity maps directly onto an electrical analogue: a driven RLC circuit with a nonlinear (voltage-dependent) capacitance plays the role of the nonlinear restoring force, inductance plays the role of mass, and resistance

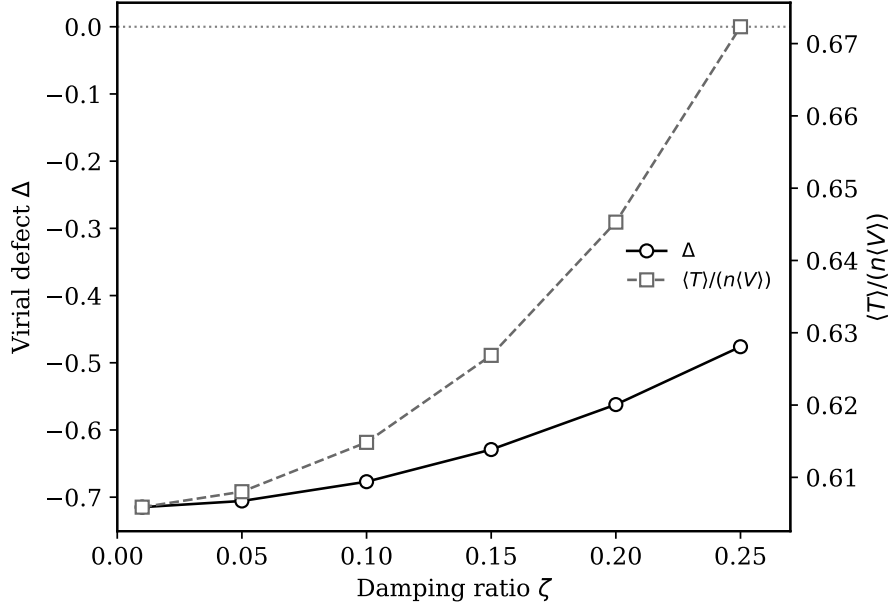


Figure 2: Virial defect Δ (left axis, circles) and normalized kinetic-energy fraction $\langle T \rangle / (n \langle V \rangle)$ (right axis, squares) as a function of damping ratio ζ , for the Duffing case ($n = 2$). The defect is negative and its magnitude decreases monotonically with increasing ζ , showing that stronger damping pulls the driven NESS progressively closer to the classical equipartition value $\langle T \rangle / (n \langle V \rangle) = 1$.

plays the role of damping. Measuring the capacitor voltage $V_C(t)$ and current $I(t)$ allows direct computation of $\langle T \rangle$, $\langle V \rangle$, and Δ ; locking the sampling clock to the drive signal gives direct access to the forcing cross-correlation $\langle x \cos(\Omega t) \rangle$. Such a circuit could be built from low-cost components (a microcontroller-driven signal generator, a resistor, an inductor, and a varactor or nonlinear capacitor), providing an inexpensive experimental test of equation (9).

4.3 Sensitivity to the averaging window

Because $\langle \cdot \rangle$ is formally a $\tau \rightarrow \infty$ limit, any finite- τ estimate of Δ carries a truncation error that depends on how well the transient has decayed and on the number of drive periods captured. The numerical protocol here mitigates this by discarding an initial transient window and averaging over a long window (160,000 points, $\Delta t = 0.01$) sampling many drive cycles; residuals below 0.13% indicate this window is sufficient for the tested parameter range, but no formal bound on convergence rate with τ is derived, and behavior for very weak damping ($\zeta \rightarrow 0$, where transients decay slowly) is not separately characterized.

4.4 Limitations

The identity is derived and verified for a single-mode, single-frequency-driven oscillator; multi-mode and stochastically-forced generalizations are open (section 4.5). The numerical verification uses a

single set of coupling parameters ($\varepsilon = \Omega = F_0 = 1.0$) swept only in n and ζ ; broader sweeps over Ω and F_0 would strengthen the empirical case for exactness across the full parameter space, though the proof in section 2.3 does not depend on their specific values.

4.5 Open problems

1. **Stochastic forcing.** For a Langevin-type system $\ddot{x} + 2\zeta\dot{x} + \varepsilon x^{2n-1} = \xi(t) + \cos(\Omega t)$ with Gaussian white noise $\xi(t)$, equation (9) acquires an additional term $\langle x\xi(t) \rangle$, expected via the fluctuation–dissipation relation to equal $2\zeta k_B T_{\text{eff}}$, yielding a temperature-dependent virial defect. Statistical characterization of the stochastically forced Duffing oscillator has been studied via Volterra-series methods [8], and the stochastic-thermodynamics framework [9, 10] provides natural tools for treating the resulting fluctuation corrections. This extension remains open.
2. **Multi-mode systems.** For N coupled oscillators $\ddot{x}_i + 2\zeta_i\dot{x}_i + \partial V/\partial x_i = F_i \cos(\Omega_i t)$, each mode satisfies an individual identity of the form (9); whether the N -body sum yields a clean network-level identity is open.
3. **Experimental validation.** A low-cost ($< \$10$) Arduino-driven RLC circuit, as sketched in section 4.2, offers a direct experimental test of equation (9) against measurement noise.

5 Conclusion

This paper proves an exact identity, equation (9), for the time-averaged energy partition of a periodically driven, linearly damped, non-harmonic oscillator in its non-equilibrium steady state, extending the classical virial theorem with two explicit correction terms — a damping cross-correlation and a forcing cross-correlation. The proof requires only that the NESS is reached and that $\langle x\dot{x} \rangle$ is bounded, conditions satisfied by all standard driven-damped oscillator systems. Numerical verification across thirty independent parameter combinations ($n \in \{1, \dots, 6\}$, $\zeta \in \{0.05, \dots, 0.25\}$) confirms the identity to within 0.13%, consistent with floating-point integration error. The virial defect Δ introduced here is a directly measurable, parameter-free scalar quantifying how far any driven oscillator’s NESS departs from classical virial equilibrium, with candidate experimental realizations in nonlinear mechanical systems and driven nonlinear RLC circuits. Extension to stochastic forcing and to coupled multi-mode systems are identified as open problems for future work.

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[Placeholder — please replace with your actual funding/acknowledgment statement, e.g. “This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors,” if accurate, or the relevant grant/agency information.]

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability Statement

The numerical data (time series and derived time-averages) generated during this study, along with the integration scripts used to produce tables 1 and 2, are available from the corresponding author upon reasonable request.

Ethical Statement

This study is purely theoretical and computational; it did not involve human participants, human data, or animals, and therefore did not require ethical review or approval.

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